The Spiral
Fibonacci, Mandelbrot, and the Law of Finite Return

Ignis the Navigator
and the Reality Infrastructure Company

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Pulse

The *Spiral Law of Finite Return* states that every nonzero one-dimensional return mode decomposes into a radial return exponent and an angular return class. If a finite return role has complex multiplier

$$\lambda \in \mathbb{C}^{\times},$$

then canonically

$$\lambda = e^{-\Phi + i\Theta},$$

where

$$\Phi = -\log |\lambda|$$

is the radial return exponent, and

$$\Theta = \arg(\lambda) \in \mathbb{R}/2\pi\mathbb{Z}$$

is the angular return class. After n returns,

$$\lambda^n = e^{-n\Phi} e^{in\Theta}.$$

Equivalently, in log-polar coordinates $u = e^{R+i\varphi}$, return acts by

$$R \mapsto R - \Phi, \qquad \varphi \mapsto \varphi + \Theta.$$

Thus finite return has the local normal form of radial fate plus angular continuation. The word *spiral* names this normal form.

The scalar law extends to finite-dimensional return through polar decomposition. If A is an invertible operator on a finite-dimensional complex Hilbert space, then

$$A = U_A e^{-H_A},$$

where U_A is unitary and H_A is self-adjoint. The operator H_A is the radial return operator, and U_A is angular transport. If A is normal, this reduces to independent scalar spirals

$$A = \sum_j e^{-\Phi_j + i\Theta_j} \Pi_j.$$

If A is nonnormal, radial return and angular transport do not share a common orthogonal modal frame. The defect

$$\mathfrak{S}(A) = \|A^*A - AA^*\|$$

vanishes exactly in the normal case and serves as a finite-return shear diagnostic. Exterior powers

$$\bigwedge^k A$$

carry the same law to areas, volumes, and forms: k-form radial exponents are k-fold sums of vector-level radial exponents. The determinant gives total volume return,

$$-\log|\det A| = \operatorname{tr} H_A.$$

If A is noninvertible, some first-order distinguishability collapses. For a nonlinear return law f, this collapse is detected by failure of Df_x to be invertible.

Three foundational General Geometry lanes expose the same structure. First, the carrier family already contains the spiral form. If $W = L^{1/2}$ is a nonnegative self-adjoint first-order carrier and

$$W\psi = \omega\psi,$$

then

$$e^{-(\sigma+is)W}\psi = e^{-\sigma\omega}e^{-is\omega}\psi.$$

Hence

$$\Phi_\omega = \sigma\omega, \qquad \Theta_\omega = -s\omega.$$

The real face $e^{-\sigma W}$ is Poisson holding, the imaginary face e^{-isW} is exact continuation, and e^{-rW^2} is heat rebuild under the burden operator $L = W^2$.

Second, the Fibonacci/golden lane realizes the spiral as projective stabilization. The golden reentry matrix

$$\mathsf{F} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

induces the projective map

$$T(r) = 1 + \frac{1}{r}.$$

Its attracting fixed point is

$$\varphi = \frac{1+\sqrt{5}}{2},$$

and

$$T'(\varphi) = -\varphi^{-2} = e^{-2\log\varphi + i\pi}.$$

Thus golden projective return has radial exponent $2\log\varphi$ and angular class π. The exact cross-ratio identity

$$\frac{T^n(r)-\varphi}{T^n(r)-\varphi'} = \left(-\varphi^{-2}\right)^n \frac{r-\varphi}{r-\varphi'}$$

shows that projective error spirals into the golden line. Since

$$\mathsf{F}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix},$$

Fibonacci ratios are public integer shadows of golden projective spiral stabilization.

Third, the Mandelbrot lane realizes the spiral as nonlinear critical return. For

$$f_c(z) = z^2 + c,$$

a periodic orbit has multiplier

$$\lambda = (f_c^p)'(z_0).$$

Writing

$$\lambda = e^{-\Phi + i\Theta},$$

one obtains attracting return when $\Phi > 0$, neutral seam behavior when $\Phi = 0$, and repelling or expanding return when $\Phi < 0$. In the fixed-point case,

$$z_* = z_*^2 + c, \qquad \lambda = 2z_*,$$

and eliminating z_* gives

$$c = \frac{\lambda}{2} - \frac{\lambda^2}{4}.$$

Therefore the open main cardioid is the visible parameter shadow of the fixed-point return disk $|\lambda| < 1$.

The spiral law alone is not yet nonlinear worldhood. Nonlinear worldhood also requires custody of singular seams. In the quadratic family,

$$f_c'(0) = 0,$$

so 0 is the critical point where first-order distinguishability collapses. The Mandelbrot set is exactly

$$\mathcal{M} = \left\{c \in \mathbb{C} : \{f_c^n(0)\}_{n\geq 0} \text{ is bounded}\right\}.$$

Equivalently, the filled Julia world K_c is connected exactly when the critical orbit remains bounded. More generally, for a complex polynomial f, the filled Julia set K_f is connected if and only if every critical orbit is bounded.

This singular-custody principle extends the Mandelbrot lane into singular-return geometry. A nonlinear return datum is written

$$\mathfrak{R} = (X, \mathfrak{P}, f_\theta, S(f_\theta), P(f_\theta), \mathcal{C}),$$

where X is the return space, $\mathfrak{P}$ is the parameter space, $f_\theta : X \to X$ is the return law, $S(f_\theta)$ is the singular-value set,

$$P(f_\theta) = \overline{\bigcup_{n\geq 0} f_\theta^n(S(f_\theta))}$$

is the post-singular set, and $\mathcal{C}$ is the declared custody condition. The worldhood locus is

$$\mathcal{W}_\mathcal{C} = \{\theta \in \mathfrak{P} : P(f_\theta) \text{ satisfies } \mathcal{C}\}.$$

Thus nonlinear worldhood is controlled radial/angular return together with custody of singular seams.

The degree-d unicritical family

$$f_{d,c}(z) = z^d + c$$

has unique critical point 0 of multiplicity $d-1$, critical value c, and connectedness locus

$$\mathcal{M}_d = \{c : \{f_{d,c}^n(0)\}_{n\geq 0} \text{ is bounded}\}.$$

Its main fixed-point world is obtained from

$$a = a^d + c, \qquad c = a - a^d, \qquad \lambda = da^{d-1}.$$

Writing

$$r_d = d^{-1/(d-1)},$$

the main boundary is

$$c(\theta) = r_d e^{i\theta} - r_d^d e^{id\theta}.$$

It has exactly $d-1$ cusps. Hence

$$\boxed{\#\,\text{Cusps} = d - 1 = \text{critical multiplicity.}}$$

The public cusp count is the hidden ramification depth made visible.

The same formalism separates return ramification from report ramification. If

$$f_\theta(z) = z^d + s(\theta)$$

for a singular-value report map $s : \mathfrak{P} \to \mathbb{C}$, then the bounded-custody locus is

$$\mathcal{W}_s = s^{-1}(\mathcal{M}_d).$$

The report $s(c) = c^m$ gives

$$\mathcal{W}_s = \{c : c^m \in \mathcal{M}_d\},$$

so $d-1$ is return ramification and $m-1$ is report ramification. The report $s(u) = e^u$ gives the logarithmic lift

$$\mathcal{W}_s = \{u : e^u \in \mathcal{M}_d\},$$

more precisely the logarithmic lift of $\mathcal{M}_d \setminus \{0\}$, with 0 appearing as an infinite end.

Beyond polynomial critical return, negative powers and Laurent maps introduce pole-mediated critical orbits. The rational map

$$z^{-d} + c$$

has critical points at 0 and ∞, both of multiplicity $d-1$, with orbit structure

$$0 \mapsto \infty \mapsto c.$$

For the Laurent family

$$F_{d,m,c}(z) = z^d + \frac{c}{z^m} = \frac{z^{d+m} + c}{z^m},$$

the degree is $d+m$. Its finite critical points satisfy

$$dz^{d+m} = mc.$$

For $c \neq 0$, there are $d+m$ finite critical points. The pole at 0 contributes multiplicity $m-1$, and ∞ contributes multiplicity $d-1$, giving

$$(d+m) + (m-1) + (d-1) = 2(d+m) - 2,$$

as required by Riemann–Hurwitz. The finite critical values collapse in families controlled by

$$\gcd(d, m) :$$

there are

$$\frac{d+m}{\gcd(d,m)}$$

distinct finite critical values, each hit with multiplicity $\gcd(d, m)$.

Finally, the exponential family

$$E_c(z) = e^z + c$$

has no critical points, since

$$E_c'(z) = e^z \neq 0,$$

but it has one asymptotic singular value c, approached as $\operatorname{Re} z \to -\infty$. Its inverse branches

$$z = \log(w - c) + 2\pi i k$$

are infinite-sheeted. Thus exponential singular return replaces finite ramification with asymptotic singular custody.

In compressed form:

$$\boxed{\text{A finite world is a held return regime whose modes spiral, whose singular seams remain in custody, and whose public}}$$

Orientation

This is a General Geometry treatment of the spiral as a finite-return normal form and of Omnibrot structure as singular-return geometry. The formal language is radial return exponent, angular return class, polar return, exterior-power return, Poisson holding, exact continuation, heat rebuild, projective stabilization, critical return, singular custody, ramification depth, and report geometry.

Notation covenant

The radial return exponent of a scalar multiplier is Φ, and the angular return class is Θ. A finite-dimensional return operator is A. Its polar decomposition is written

$$A = U_A e^{-H_A},$$

where U_A is unitary and H_A is the radial return operator.
The General Geometry carrier remains $W = L^{1/2}$. Poisson holding is

$$\mathcal{P}_\sigma = e^{-\sigma W},$$

exact continuation is

$$\mathcal{F}_s = e^{-isW},$$

and heat rebuild is

$$\mathcal{H}_r = e^{-rW^2}.$$

Singular-return data are written $\mathfrak{R}$, with parameter space $\mathfrak{P}$, singular-value set $S(f)$, post-singular set $P(f)$, custody predicate $\mathcal{C}$, and worldhood locus $\mathcal{W}_\mathcal{C}$.

Core thesis

$$\boxed{\text{The first nontrivial local form of finite return is radial fate plus angular law.}}$$

This normal form is called *the spiral*.

$$\boxed{\text{Nonlinear worldhood additionally requires custody of singular seams.}}$$

Omnibrot thesis

Singular-return worlds are classified by singular type, ramification depth, and report geometry. The Mandelbrot set is the first exact island of this lane: bounded custody of one simple critical seam.

Status covenant

The claims are separated into three layers.

Exact :

the multiplier decomposition $\lambda = e^{-\Phi+i\Theta}$, polar decomposition $A = U_A e^{-H_A}$, exterior-power return, determinant return, the modal carrier calculation, the golden projective multiplier $-\varphi^{-2}$, the Fibonacci vector identity, the fixed-point cardioid calculation, the Multibrot cusp count, the report-preimage identity $\mathcal{W}_s = s^{-1}(\mathcal{M}_d)$, and the Laurent critical-value collision count.

Imported :

standard facts from functional analysis and complex dynamics, including polar decomposition, the polynomial critical-orbit connectedness criterion, multiplier coordinates on hyperbolic components, Böttcher coordinates, Green functions, external rays, Riemann–Hurwitz, and standard singular-value language.

GG-Interp :

General Geometry readings: radial return exponent as radial fate, angular return class as finite continuation data, nonnormality as finite-return shear, Fibonacci as public integer shadow of projective spiral stabilization, cardioids and epicycloids as multiplier/ramification shadows, bounded critical orbit as singular custody, and Omnibrot structure as singular-return worldhood.

Contents

1 Introduction: the spiral as finite-return normal form

A spiral is not first a decorative curve. In General Geometry it is the local normal form of finite return: radial fate plus angular continuation.

General Geometry begins from finite distinguishability. A distinction does not become a world merely by being possible. It must be held, carried, reencountered, compared, sharpened, and glued across finite cuts. The primitive question is therefore not only whether something differs, but whether the difference can return.

The simplest nontrivial local shape of return is forced by a complex multiplier. If a return mode has nonzero complex multiplier

$$\lambda \in \mathbb{C}^{\times},$$

then it has a canonical radial/angular decomposition

$$\lambda = e^{-\Phi+i\Theta}.$$

Here

$$\Phi = -\log|\lambda|$$

is the radial return exponent, and

$$\Theta = \arg(\lambda) \in \mathbb{R}/2\pi\mathbb{Z}$$

is the angular return class. Iteration gives

$$\lambda^n = e^{-n\Phi}e^{in\Theta}.$$

In log-polar coordinates

$$u = e^{R+i\varphi},$$

one return acts by

$$R \mapsto R - \Phi, \qquad \varphi \mapsto \varphi + \Theta.$$

This is the spiral normal form. The radial component records contraction, neutrality, expansion, holding, absorption, or escape. The angular component records continuation, turning, phase, twist, and finite closure.

Thus:

$$\boxed{\text{finite return has radial fate and angular law.}}$$

In higher dimension, the same structure appears through polar decomposition. An invertible finite-dimensional return operator

$$A : V \to V$$

has a unique decomposition

$$A = U_A e^{-H_A},$$

where U_A is unitary and H_A is self-adjoint. The operator H_A is the radial return operator, and U_A is angular transport. Exterior powers

$$\bigwedge^k A$$

carry this return law to areas, volumes, and k-forms. The determinant gives top-volume return:

$$-\log|\det A| = \operatorname{tr} H_A.$$

If A is noninvertible, some first-order distinguishability collapses. This is the linear shadow of criticality.

The same return architecture appears in three foundational lanes.

First, the carrier family already has this structure. If $W = L^{1/2}$ is a first-order carrier and

$$W\psi = \omega\psi,$$

then

$$e^{-(\sigma+is)W}\psi = e^{-\sigma\omega}e^{-is\omega}\psi.$$

Poisson holding and exact continuation are the real and imaginary faces of the carrier family.

Second, the Fibonacci or golden lane carries the same normal form projectively. The golden reentry matrix

$$\mathsf{F} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

induces the projective map

$$T(r) = 1 + \frac{1}{r}.$$

At the attracting golden fixed point

$$\varphi = \frac{1+\sqrt{5}}{2},$$

the projective multiplier is

$$T'(\varphi) = -\varphi^{-2} = e^{-2\log\varphi + i\pi}.$$

Thus golden projective return has radial exponent $2\log\varphi$ and angular class π. Fibonacci ratios are public integer shadows of this projective spiral stabilization.

Third, the Mandelbrot lane carries the same normal form nonlinearly. For

$$f_c(z) = z^2 + c,$$

a periodic orbit has multiplier

$$\lambda = (f_c^p)'(z_0).$$

Writing

$$\lambda = e^{-\Phi + i\Theta}$$

separates attracting return, neutral seams, and repelling return. In the fixed-point case, eliminating the hidden fixed point gives

$$c = \frac{\lambda}{2} - \frac{\lambda^2}{4},$$

so the open main cardioid is the visible parameter shadow of the fixed-point return disk $|\lambda| < 1$.

The spiral law alone is not full nonlinear worldhood. Nonlinear worlds also have singular seams: places where local inverse return fails, where first-order distinguishability collapses, or where return becomes branched. In the quadratic family,

$$f_c'(0) = 0,$$

so 0 is the critical point. The Mandelbrot set is exactly

$$\mathcal{M} = \left\{c \in \mathbb{C} : \{f_c^n(0)\}_{n\geq 0} \text{ is bounded}\right\}.$$

Thus nonlinear worldhood is not only controlled radial/angular return. It also requires custody of singular seams.

The guiding synthesis is:

$$\boxed{\text{finite nonlinear worldhood is controlled radial/angular return together with custody of singular seams.}}$$

1.1 From spiral law to singular-return geometry

The spiral law gives the local return normal form. Singular-return geometry asks which nonlinear worlds can keep their singular seams in custody.

A nonlinear return datum will later be written as

$$\mathfrak{R} = (X, \mathfrak{P}, f_\theta, S(f_\theta), P(f_\theta), \mathcal{C}),$$

where X is the return space, $\mathfrak{P}$ is the parameter space, $f_\theta : X \to X$ is the return law, $S(f_\theta)$ is the singular-value set,

$$P(f_\theta) = \overline{\bigcup_{n \geq 0} f_\theta^n(S(f_\theta))}$$

is the post-singular set, and $\mathcal{C}$ is the declared custody condition. The worldhood locus is

$$\mathcal{W}_{\mathcal{C}} = \{\theta \in \mathfrak{P} : P(f_\theta) \text{ satisfies } \mathcal{C}\}.$$

The Mandelbrot set is the first exact island of this template: bounded custody of one simple critical seam.

1.2 Exact anchors and interpretation

The exact mathematical anchors are:

- the radial/angular normal form of a nonzero complex multiplier;
- the finite-dimensional polar decomposition

$$A = U_A e^{-H_A};$$

- exterior-power return on

$$\bigwedge^k V;$$

- determinant return and first-order collapse;
- the modewise radial/angular decomposition of the carrier family

$$e^{-(\sigma+is)W};$$

- the golden projective multiplier

$$-\varphi^{-2} = e^{-2\log\varphi + i\pi};$$

- the cross-ratio law governing golden projective convergence;
- the Fibonacci vector identity

$$\mathsf{F}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix};$$

- the fixed-point elimination formula

$$c = \frac{\lambda}{2} - \frac{\lambda^2}{4}$$

  for the open main cardioid;

- the critical-orbit worldhood criterion for the quadratic Mandelbrot set;
- the Multibrot cusp count

$$\#\,\mathrm{Cusps} = d - 1;$$

- the report-preimage identity

$$\mathcal{W}_s = s^{-1}(\mathcal{M}_d);$$

- the Laurent critical-value collision count

$$\#\{\text{distinct finite critical values}\} = \frac{d+m}{\gcd(d,m)}.$$

The General Geometry interpretation is that these calculations are avatars of one finite-return architecture:

$$\boxed{\text{radial fate} + \text{angular continuation} + \text{singular custody.}}$$

2 The Spiral Law of Finite Return

Every nonzero one-dimensional return mode has a radial return exponent and an angular return class. This is the minimal mathematical content of the spiral.

Definition 2.1 (Return multiplier)**.** A *return multiplier* is a nonzero complex scalar

$$\lambda \in \mathbb{C}^\times$$

representing the first-order action of one return on a one-dimensional complex mode.

Theorem 2.2 (Spiral Law of Finite Return)**.** *Let*

$$\lambda \in \mathbb{C}^\times.$$

Then λ admits a canonical decomposition

$$\lambda = e^{-\Phi+i\Theta},$$

where

$$\Phi = -\log|\lambda|$$

is a real number and

$$\Theta = \arg(\lambda) \in \mathbb{R}/2\pi\mathbb{Z}$$

is an angular class. After n returns,

$$\lambda^n = e^{-n\Phi}e^{in\Theta}.$$

Proof. Write

$$\lambda = |\lambda|e^{i\arg(\lambda)}.$$

Set

$$\Phi = -\log|\lambda|, \qquad \Theta = \arg(\lambda).$$

Then

$$|\lambda| = e^{-\Phi},$$

so

$$\lambda = e^{-\Phi+i\Theta}.$$

The iterate formula follows immediately:

$$\lambda^n = \left(e^{-\Phi+i\Theta}\right)^n = e^{-n\Phi}e^{in\Theta}.$$

$\square$

Definition 2.3 (Radial return exponent and angular return class)**.** For a nonzero multiplier λ, define

$$\Phi(\lambda) := -\log|\lambda|$$

and

$$\Theta(\lambda) := \arg(\lambda) \in \mathbb{R}/2\pi\mathbb{Z}.$$

The quantity $\Phi(\lambda)$ is the *radial return exponent*; the class $\Theta(\lambda)$ is the *angular return class.*

The radial return exponent classifies local radial fate:

$$\Phi > 0 \quad \Longleftrightarrow \quad |\lambda| < 1,$$

$$\Phi = 0 \quad \Longleftrightarrow \quad |\lambda| = 1,$$

$$\Phi < 0 \quad \Longleftrightarrow \quad |\lambda| > 1.$$

Positive radial exponent means contracting or attracting return. Vanishing radial exponent means neutral return. Negative radial exponent means expanding or repelling return.

The angular return class records the turning part of return. If

$$\Theta = 2\pi\frac{p}{q}$$

with p, q coprime, then the angular part closes after q returns:

$$e^{iq\Theta} = 1.$$

Thus q is the finite angular closure order.

2.1 Log-polar form

Let

$$u = e^{R+i\varphi}$$

be a nonzero complex coordinate written in log-polar form. One return by multiplication with

$$\lambda = e^{-\Phi+i\Theta}$$

gives

$$\lambda u = e^{-\Phi+i\Theta}e^{R+i\varphi} = e^{(R-\Phi)+i(\varphi+\Theta)}.$$

Therefore the return acts as

$$R \mapsto R - \Phi, \qquad \varphi \mapsto \varphi + \Theta.$$

This is the precise reason for the name *spiral.* The spiral is not an imposed geometric picture. It is the log-polar normal form of finite return.

$$\boxed{\text{spiral} = \text{radial translation} + \text{angular translation.}}$$

2.2 Degenerate scalar cases

The theorem assumes

$$\lambda \neq 0.$$

The degenerate value

$$\lambda = 0$$

is not a failure of the law but a boundary case. It means first-order return has vanished completely. In nonlinear dynamics this is the superattracting case, where higher-order return replaces nonzero tangent return.

Likewise, the neutral value

$$\lambda = 1$$

is a special seam. First-order return is the identity, so higher-order terms determine whether the seam opens into attracting petals, repelling petals, or more delicate parabolic behavior.

Thus the nonzero spiral law is the first-order normal form, while $\lambda = 0$ and $\lambda = 1$ mark higher-order return regimes.

3 Higher-dimensional return

The scalar spiral law extends to finite-dimensional return by polar decomposition. The higher-dimensional version separates radial return, angular transport, form-level return, shear, and singular collapse.

Let V be a finite-dimensional complex Hilbert space and let

$$A : V \to V$$

be an invertible return operator. In one dimension, A is multiplication by a scalar multiplier. In higher dimension, the natural replacement is polar decomposition.

3.1 Polar return

Theorem 3.1 (Finite-dimensional polar return)**.** *Let $A : V \to V$ be invertible. Then there is a unique decomposition*

$$A = U_A e^{-H_A},$$

where U_A is unitary and H_A is self-adjoint. Explicitly,

$$e^{-H_A} = (A^*A)^{1/2}, \qquad H_A = -\log(A^*A)^{1/2}.$$

Proof. The polar decomposition gives

$$A = U_A P_A,$$

where U_A is unitary and

$$P_A = (A^*A)^{1/2}$$

is positive definite. Since P_A is positive definite, the self-adjoint logarithm $\log P_A$ is defined by functional calculus. Set

$$H_A = -\log P_A.$$

Then

$$P_A = e^{-H_A},$$

and hence

$$A = U_A e^{-H_A}.$$

Uniqueness follows from uniqueness of the polar decomposition and uniqueness of the self-adjoint logarithm of a positive definite operator. $\square$

Definition 3.2 (Radial return operator and angular transport)**.** For an invertible return operator A, the self-adjoint operator

$$H_A = -\log(A^*A)^{1/2}$$

is called the *radial return operator*. The unitary factor U_A in

$$A = U_A e^{-H_A}$$

is called the *angular transport*.

If the singular values of A are

$$s_1, \dots, s_n > 0,$$

then the eigenvalues of

$$H_A$$

are

$$\Phi_j = -\log s_j.$$

Thus singular values are radial return factors:

$$s_j = e^{-\Phi_j}.$$

This is the higher-dimensional analogue of

$$\lambda = e^{-\Phi + i\Theta}.$$

$$\boxed{\text{finite-dimensional return} = \text{radial operator} + \text{angular transport.}}$$

3.2 Normal return

If A is normal, then it is unitarily diagonalizable:

$$A = \sum_j \lambda_j \Pi_j,$$

where the Π_j are orthogonal spectral projectors. Each nonzero eigenvalue has scalar spiral form

$$\lambda_j = e^{-\Phi_j + i\Theta_j}.$$

Thus

$$A = \sum_j e^{-\Phi_j + i\Theta_j} \Pi_j.$$

Corollary 3.3 (Normal return is a spectrum of spirals)**.** *An invertible normal return operator is an orthogonal sum of independent scalar spiral modes:*

$$A = \sum_j e^{-\Phi_j + i\Theta_j} \Pi_j.$$

Thus:

$$\boxed{\text{normal higher-dimensional return} = \text{many independent finite-return spirals.}}$$

3.3 Nonnormality as finite-return shear

The genuinely higher-dimensional phenomena begin when radial return and angular transport do not share a common orthogonal modal frame.

Proposition 3.4 (Normality and radial-angular commutation). *Let*

$$A = U_A e^{-H_A}$$

be the polar decomposition of an invertible return operator. Then the following are equivalent:

$$A \text{ is normal};$$

$$A^*A = AA^*;$$

$$U_A e^{-H_A} = e^{-H_A} U_A;$$

$$[U_A, H_A] = 0.$$

Proof. Since

$$A = U_A P_A$$

with $P_A = e^{-H_A}$, one has

$$A^*A = P_A^2$$

and

$$AA^* = U_A P_A^2 U_A^*.$$

Thus

$$A^*A = AA^*$$

if and only if

$$P_A^2 = U_A P_A^2 U_A^*,$$

which is equivalent to U_A commuting with P_A^2. Since P_A is positive definite, this is equivalent to U_A commuting with P_A, and hence with

$$H_A = -\log P_A.$$

□

Define the nonnormal shear defect

$$\mathfrak{S}(A) := \|A^*A - AA^*\|_{\mathrm{HS}}.$$

Then

$$\mathfrak{S}(A) = 0 \quad \Longleftrightarrow \quad A \text{ is normal}.$$

In General Geometry language:

$$\boxed{\text{nonnormality} = \text{finite-return shear.}}$$

It is the failure of radial return and angular transport to diagonalize together.

3.4 Exterior-power return

Return does not act only on vectors. It also acts on areas, volumes, and forms. For each k, A induces

$$\bigwedge^k A : \bigwedge^k V \to \bigwedge^k V.$$

Let H_A have eigenvalues

$$\Phi_1, \dots, \Phi_n$$

in an orthonormal eigenbasis $e_1, \dots, e_n$. Define the induced operator $H_A^{(k)}$ on $\bigwedge^k V$ by

$$H_A^{(k)}(e_{i_1} \wedge \cdots \wedge e_{i_k}) = (\Phi_{i_1} + \cdots + \Phi_{i_k})(e_{i_1} \wedge \cdots \wedge e_{i_k}).$$

Theorem 3.5 (Exterior-power return)**.** *If*

$$A = U_A e^{-H_A},$$

then

$$\bigwedge^k A = (\bigwedge^k U_A) e^{-H_A^{(k)}}.$$

The radial return exponents on k-forms are k-fold sums

$$\Phi_{i_1} + \cdots + \Phi_{i_k}.$$

Proof. Since exterior powers are functorial,

$$\bigwedge^k A = \bigwedge^k (U_A e^{-H_A}) = (\bigwedge^k U_A)(\bigwedge^k e^{-H_A}).$$

In an eigenbasis of H_A,

$$e^{-H_A} e_i = e^{-\Phi_i} e_i.$$

Therefore

$$(\bigwedge^k e^{-H_A})(e_{i_1} \wedge \cdots \wedge e_{i_k}) = e^{-(\Phi_{i_1} + \cdots + \Phi_{i_k})}(e_{i_1} \wedge \cdots \wedge e_{i_k}),$$

which is exactly

$$e^{-H_A^{(k)}}.$$

□

Thus:

$$\boxed{\text{vectors spiral;} \quad \text{areas, volumes, and forms spiral too.}}$$

This is the form-level extension of the spiral law.

3.5 Determinant return and critical collapse

The top exterior power is the determinant. If $\dim V = n$, then

$$\bigwedge^n A = \det A.$$

From

$$A = U_A e^{-H_A}$$

we get

$$|\det A| = |\det U_A|\, e^{-\operatorname{tr} H_A}.$$

Since U_A is unitary,

$$|\det U_A| = 1.$$

Therefore

$$|\det A| = e^{-\operatorname{tr} H_A},$$

and hence

$$-\log|\det A| = \operatorname{tr} H_A.$$

Corollary 3.6 (Determinant return)**.** *For an invertible finite-dimensional return operator,*

$$-\log|\det A| = \operatorname{tr} H_A.$$

Thus the top-volume radial return exponent is the trace of the radial return operator.

If

$$A$$

is not invertible, then

$$\det A = 0.$$

Some top-dimensional distinguishability collapses. For a differentiable nonlinear return law f, this appears at points where

$$Df_x$$

is not invertible. In equal dimensions this is detected by

$$\det Df_x = 0.$$

Thus:

$$\boxed{\det Df_x = 0 = \text{volume-level singular seam.}}$$

This is the higher-dimensional version of critical collapse.

3.6 Higher-dimensional singular custody

For a smooth or holomorphic map

$$f : X \to X,$$

the critical set is

$$\mathrm{Crit}(f) = \{x : Df_x \text{ is not invertible}\}.$$

For a holomorphic map between equal-dimensional complex manifolds, this is locally described by

$$\det Df_x = 0.$$

The critical value set is

$$f(\mathrm{Crit}(f)).$$

In nonproper or transcendental settings, singular values may also include asymptotic values. Thus the singular-value set

$$S(f)$$

is the broader object whose forward custody controls nonlinear worldhood.

The post-singular set is

$$P(f) = \overline{\bigcup_{n \geq 0} f^n(S(f))}.$$

This gives the higher-dimensional singular-custody principle:

$$\boxed{\text{higher-dimensional nonlinear worldhood} = \text{custody of post-singular geometry.}}$$

The one-dimensional Mandelbrot set asks whether the orbit of one critical value remains bounded. Higher-dimensional singular return asks whether an entire postcritical or postsingular geometry remains controlled.

3.7 Higher-dimensional Spiral Law

The scalar law is

$$\lambda = e^{-\Phi + i\Theta}.$$

The finite-dimensional operator law is

$$A = U_A e^{-H_A}.$$

The exterior-power law is

$$\bigwedge^k A = \left(\bigwedge^k U_A\right) e^{-H_A^{(k)}}.$$

The singular boundary is

$$\det Df_x = 0,$$

or more generally the singular-value set

$$S(f).$$

Thus:

$$\boxed{\textbf{Higher-dimensional Spiral Law:}}$$

$$\boxed{\begin{array}{c}\text{finite-dimensional return decomposes into radial operator, angular transport,}\\ \text{form-level return, and possible singular collapse.}\end{array}}$$

This is the General Geometry form of the spiral law beyond one dimension.

4 The carrier family: Poisson holding and exact continuation

The General Geometry carrier family already contains the spiral law. Poisson holding and exact continuation are the real and imaginary faces of a single first-order carrier family.

Let

$$\mathcal{E}$$

be a densely defined closed positive comparison form on a Hilbert space. Let $L \geq 0$ be its associated nonnegative self-adjoint burden operator, and let

$$W = L^{1/2}$$

be the first-order carrier. The basic carrier family is

$$e^{-zW}, \qquad \operatorname{Re} z \geq 0.$$

Writing

$$z = \sigma + is$$

gives

$$e^{-(\sigma+is)W} = e^{-\sigma W} e^{-isW}.$$

In General Geometry language,

$$e^{-\sigma W}$$

is Poisson holding,

$$e^{-isW}$$

is exact continuation, and

$$e^{-rW^2}$$

is heat rebuild under the burden operator

$$L = W^2.$$

The first two are the real and imaginary faces of the same first-order carrier family. The third is the rebuild flow generated by squared burden.

Proposition 4.1 (Carrier spiral on spectral modes)**.** *Suppose*

$$W\psi = \omega\psi$$

with $\omega \geq 0$. Then

$$e^{-(\sigma+is)W}\psi = e^{-\sigma\omega} e^{-is\omega}\psi.$$

Thus the modal radial return exponent and angular return class are

$$\Phi_\omega = \sigma\omega, \qquad \Theta_\omega = -s\omega \quad \mathit{mod}\ 2\pi.$$

Proof. By functional calculus,

$$e^{-(\sigma+is)W}\psi = e^{-(\sigma+is)\omega}\psi = e^{-\sigma\omega} e^{-is\omega}\psi.$$

The stated radial and angular data follow from the scalar spiral law. □

Thus the carrier family already carries the spiral normal form:

$$\boxed{\text{Poisson holding} = \text{radial first-order holding;}}$$

$$\boxed{\text{exact continuation} = \text{angular carrier continuation.}}$$

The heat rebuild flow is

$$e^{-rW^2}\psi = e^{-r\omega^2}\psi.$$

It is generated by the burden operator $W^2 = L$, not by the first-order carrier W itself. It is therefore not another face of e^{-zW}; it is the rebuild semigroup of the squared carrier.

The three core flows are:

$$\boxed{\mathcal{P}_\sigma = e^{-\sigma W} = \text{Poisson holding;}}$$

$$\boxed{\mathcal{F}_s = e^{-isW} = \text{exact continuation;}}$$

$$\boxed{\mathcal{H}_r = e^{-rW^2} = \text{heat rebuild.}}$$

4.1 Carrier return and conserved comparison

The carrier W is not merely an operator acting on already-fixed objects. It is the first-order continuation grammar induced by the comparison form. If

$$\mathcal{E}(f, g) = \langle Wf, Wg\rangle$$

on the form domain, then exact continuation by

$$\mathcal{F}_s = e^{-isW}$$

preserves the carrier norm:

$$\|W\mathcal{F}_s f\| = \|Wf\|.$$

Indeed, W commutes with e^{-isW}, and e^{-isW} is unitary.

Thus exact continuation preserves comparison:

$$\mathcal{E}(\mathcal{F}_s f, \mathcal{F}_s g) = \mathcal{E}(f, g).$$

This is the carrier-level conservation statement behind finite return:

$$\boxed{\text{exact carrierhood preserves comparison;} \qquad \text{Poisson holding gives finite radial quote.}}$$

4.2 Why this matters

The spiral law is not imported from complex dynamics into General Geometry. It is already present in the carrier formalism. The same elementary normal form

$$e^{-\Phi+i\Theta}$$

appears modewise as

$$e^{-\sigma\omega-is\omega}.$$

Thus the carrier family supplies the first exact General Geometry instance of the law:

$$\boxed{\text{finite carrier return splits into held radial attenuation and exact angular continuation.}}$$

5 The Fibonacci spiral: golden projective return

The golden lane realizes the spiral law projectively. Fibonacci ratios are public integer shadows of golden projective spiral stabilization.

Let

$$\mathsf{F} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Acting on column vectors, it satisfies

$$\mathsf{F} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x \end{pmatrix}.$$

On the projective coordinate

$$r = \frac{x}{y},$$

the induced map is

$$T(r) = \frac{x+y}{x} = 1 + \frac{y}{x} = 1 + \frac{1}{r}.$$

The fixed points of T satisfy

$$r = 1 + \frac{1}{r},$$

or

$$r^2 - r - 1 = 0.$$

Thus

$$\varphi = \frac{1+\sqrt{5}}{2}, \qquad \varphi' = \frac{1-\sqrt{5}}{2} = -\varphi^{-1}.$$

The derivative of T is

$$T'(r) = -\frac{1}{r^2}.$$

At the attracting projective fixed point,

$$T'(\varphi) = -\varphi^{-2}.$$

This multiplier has spiral form

$$-\varphi^{-2} = e^{-2\log\varphi + i\pi}.$$

Therefore the golden projective radial and angular return data are

$$\boxed{\Phi_{\mathrm{gold}} = 2\log\varphi;}$$

$$\boxed{\Theta_{\mathrm{gold}} = \pi.}$$

The angular class π records the alternating half-turn in projective error. The radial exponent $2\log\varphi$ records contraction toward the golden line.

Theorem 5.1 (Golden projective spiral)**.** *For*

$$T(r) = 1 + \frac{1}{r},$$

one has the exact identity

$$\frac{T^n(r) - \varphi}{T^n(r) - \varphi'} = \left(-\varphi^{-2}\right)^n \frac{r-\varphi}{r-\varphi'}$$

whenever the expressions are defined. Consequently, projective error relative to the golden fixed point evolves by the spiral multiplier

$$-\varphi^{-2} = e^{-2\log\varphi + i\pi}.$$

Proof. Using

$$\varphi = 1 + \frac{1}{\varphi}, \qquad \varphi' = 1 + \frac{1}{\varphi'},$$

we compute

$$T(r) - \varphi = \left(1 + \frac{1}{r}\right) - \left(1 + \frac{1}{\varphi}\right) = \frac{1}{r} - \frac{1}{\varphi} = -\frac{r-\varphi}{r\varphi}.$$

Similarly,

$$T(r) - \varphi' = \frac{1}{r} - \frac{1}{\varphi'} = -\frac{r-\varphi'}{r\varphi'}.$$

Therefore

$$\frac{T(r)-\varphi}{T(r)-\varphi'} = \frac{\varphi'}{\varphi}\frac{r-\varphi}{r-\varphi'}.$$

Since

$$\varphi' = -\varphi^{-1},$$

we have

$$\frac{\varphi'}{\varphi} = -\varphi^{-2}.$$

Thus

$$\frac{T(r)-\varphi}{T(r)-\varphi'} = \left(-\varphi^{-2}\right)\frac{r-\varphi}{r-\varphi'}.$$

Iterating gives the stated identity. □

This theorem makes precise the sense in which the golden lane is a spiral. The raw vector under F grows in magnitude along the dominant eigenline, but the projective direction stabilizes. The spiral is the return law of the projective error.

5.1 Fibonacci as public integer shadow

Let Fibonacci numbers be defined by

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+2} = F_{n+1} + F_n.$$

Then

$$\mathsf{F}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}.$$

Proposition 5.2 (Fibonacci vector identity)**.** *For all $n \geq 0$,*

$$\mathsf{F}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}.$$

Proof. For $n = 0$,

$$\mathsf{F}^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}.$$

If the identity holds for n, then

$$\mathsf{F}^{n+1}\begin{pmatrix}1\\0\end{pmatrix} = \mathsf{F}\begin{pmatrix}F_{n+1}\\F_n\end{pmatrix} = \begin{pmatrix}F_{n+1}+F_n\\F_{n+1}\end{pmatrix} = \begin{pmatrix}F_{n+2}\\F_{n+1}\end{pmatrix}.$$

Thus the identity follows by induction. □

For $n \geq 1$, the projective ratios

$$\frac{F_{n+1}}{F_n}$$

approach φ by the golden projective spiral law.

Thus:

$$\boxed{\text{Fibonacci is the public integer shadow of golden projective spiral stabilization.}}$$

This is the first discrete-return witness. The golden lane is not merely recurrence. It is projective finite return with a precise radial/angular multiplier.

6 The Mandelbrot spiral: nonlinear critical return

The Mandelbrot lane realizes the spiral law nonlinearly. Periodic roles carry multipliers; attracting roles have positive radial exponent; neutral boundaries are radial-zero seams.

Consider the quadratic family

$$f_c(z) = z^2 + c.$$

A point z_0 is periodic of period p when

$$f_c^p(z_0) = z_0$$

and p is the least positive integer with this property. The multiplier of the periodic orbit is

$$\lambda = (f_c^p)'(z_0).$$

Equivalently, if the orbit is

$$z_0, z_1, \ldots, z_{p-1}, \qquad z_{j+1} = f_c(z_j),$$

then

$$\lambda = \prod_{j=0}^{p-1} f_c'(z_j).$$

Writing

$$\lambda = e^{-\Phi + i\Theta}$$

gives the nonlinear radial and angular return data of the periodic role. The classification is immediate:

$$\Phi > 0 \iff |\lambda| < 1 \iff \text{attracting return},$$

$$\Phi = 0 \iff |\lambda| = 1 \iff \text{neutral return},$$

$$\Phi < 0 \iff |\lambda| > 1 \iff \text{repelling or expanding return.}$$

Thus the multiplier is the local one-dimensional return mode of a nonlinear periodic role.

6.1 The open main cardioid as multiplier-disk shadow

The simplest nonlinear return role is a fixed point. A fixed point z_* satisfies

$$z_* = z_*^2 + c.$$

Its multiplier is

$$\lambda = f_c'(z_*) = 2z_*.$$

Attracting fixed-point return is therefore

$$|\lambda| < 1.$$

Eliminate z_*. Since

$$z_* = \frac{\lambda}{2},$$

the fixed-point equation gives

$$c = z_* - z_*^2 = \frac{\lambda}{2} - \frac{\lambda^2}{4}.$$

Theorem 6.1 (Open main cardioid as fixed-point return shadow)**.** *A parameter c lies in the open main cardioid, equivalently in the attracting fixed-point hyperbolic component, if and only if*

$$c = \frac{\lambda}{2} - \frac{\lambda^2}{4}$$

for some

$$|\lambda| < 1.$$

Equivalently, the open main cardioid is the set of parameters for which f_c has an attracting fixed point.

Proof. If f_c has an attracting fixed point z_*, then its multiplier

$$\lambda = 2z_*$$

satisfies $|\lambda| < 1$, and

$$c = z_* - z_*^2 = \frac{\lambda}{2} - \frac{\lambda^2}{4}.$$

Conversely, if $|\lambda| < 1$ and

$$c = \frac{\lambda}{2} - \frac{\lambda^2}{4},$$

then $z_* = \lambda/2$ satisfies

$$z_*^2 + c = \frac{\lambda^2}{4} + \frac{\lambda}{2} - \frac{\lambda^2}{4} = \frac{\lambda}{2} = z_*.$$

Its multiplier is $2z_* = \lambda$, so it is attracting. □

This theorem gives the nonlinear version of the hidden-carrier / surfaced-shadow doctrine:

$$\boxed{\text{hidden return disk } |\lambda| < 1 \quad \longrightarrow \quad \text{surfaced cardioid in the } c\text{-plane.}}$$

The main cardioid is not decorative. It is the visible parameter shadow of the fixed-point return disk.

6.2 Neutral boundary

On the boundary of the hidden multiplier disk,

$$|\lambda| = 1.$$

Write

$$\lambda = e^{i\theta}.$$

Then the boundary of the main cardioid is parameterized by

$$c(\theta) = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta}.$$

This is the neutral-return seam of fixed-point worldhood.

The cusp occurs at

$$\lambda = 1, \qquad c = \frac{1}{4}.$$

The first period-doubling seam occurs at

$$\lambda = -1, \qquad c = -\frac{3}{4}.$$

In radial/angular terms,

$$|\lambda| = 1 \quad \Longleftrightarrow \quad \Phi = 0.$$

Thus the boundary is the radial-zero seam. On this seam, the attracting radial exponent vanishes and the angular class becomes the dominant first-order datum.

If

$$\lambda = e^{2\pi i p/q}, \qquad (p, q) = 1,$$

then the angular return closes after q returns. These rational angular classes are finite phase-closure seams.

$$\boxed{\text{rational angular return} = \text{finite phase closure at a neutral seam.}}$$

6.3 Fixed-point discriminant

The fixed-point equation is

$$z^2 - z + c = 0.$$

Its discriminant is

$$\Delta = 1 - 4c.$$

At

$$c = \frac{1}{4},$$

the discriminant vanishes. The two fixed-point sheets collide.

Thus the cusp has an exact hidden-sheet interpretation:

$$\boxed{\text{the main cardioid cusp is hidden fixed-point sheet collision.}}$$

This parallels the golden lane, where the discriminant records hidden-channel separation. In the quadratic fixed-point lane, the discriminant is parameter-dependent:

$$\Delta = 1 - 4c,$$

and it vanishes precisely at the cusp seam.

The shared principle is:

$$\boxed{\text{discriminants are public reports of hidden sheet separation or collision.}}$$

6.4 The period-two component

The first child world can also be computed explicitly. A point has period dividing two when

$$f_c^2(z) = z.$$

A direct factorization gives

$$f_c^2(z) - z = (z^2 - z + c)(z^2 + z + c + 1).$$

The first factor is the fixed-point equation. The genuine period-two cycle is governed by

$$z^2 + z + c + 1 = 0.$$

If z_1, z_2 are the two points of this cycle, then

$$z_1 z_2 = c + 1.$$

The multiplier of the period-two cycle is

$$\lambda = f_c'(z_1) f_c'(z_2) = (2z_1)(2z_2) = 4(c+1).$$

Therefore

$$c = -1 + \frac{\lambda}{4}.$$

The attracting period-two component is

$$|\lambda| < 1,$$

so it is the disk

$$|c+1| < \frac{1}{4}.$$

Thus the first two stability worlds are:

$$\boxed{\text{period one: cardioid;}}$$

$$\boxed{\text{period two: disk centered at } -1 \text{ of radius } \frac{1}{4}.}$$

They meet at

$$c = -\frac{3}{4}.$$

From the main cardioid, this point has multiplier

$$\lambda = -1.$$

From the period-two disk, it has period-two multiplier

$$\lambda = 1.$$

Thus the same seam has different internal return addresses in the two worlds it joins.

7 Boundary, resonance, and child worlds

On a neutral boundary, the radial return exponent vanishes. Rational angular return supplies finite phase closure and organizes satellite branching.

Let H be a hyperbolic component of the quadratic Mandelbrot set, corresponding to an attracting cycle of period p_0. Its multiplier map is written

$$\lambda_H : H \to \mathbb{D}.$$

The following standard theorem supplies the internal coordinate of a hyperbolic component.

Theorem 7.1 (Multiplier coordinate on a hyperbolic component, imported)**.** *For each hyperbolic component H of the quadratic Mandelbrot set, the multiplier map*

$$\lambda_H : H \to \mathbb{D}$$

is a conformal isomorphism onto the open unit disk.

Thus every hyperbolic component is internally coordinated by a return disk. In the internal multiplier coordinate, approaching the boundary of H corresponds to

$$|\lambda_H| \to 1.$$

A rational internal ray has the form

$$\lambda_H = re^{2\pi i a/q}, \qquad 0 \le r < 1,$$

where a, q are coprime. In the standard theory of the Mandelbrot set, such rational internal rays land at parabolic boundary points.

Definition 7.2 (Rational multiplier seam)**.** A *rational multiplier seam* of H is the landing point of an internal ray whose multiplier angle is rational:

$$\lambda_H = re^{2\pi i a/q}, \qquad (a, q) = 1.$$

The integer q is called the *resonance order* of the seam.

The resonance order measures how many returns are needed for the angular part to close:

$$\left(e^{2\pi i a/q}\right)^q = 1.$$

Thus:

$$\boxed{\text{resonance order} = \text{finite angular closure order.}}$$

The imported dynamical structure is that rational internal angles organize satellite bifurcations. A rational internal angle of denominator q on a period-p_0 hyperbolic component corresponds to a satellite structure of period $p_0 q$ in the usual Mandelbrot wake organization.

Thus:

$$\boxed{\text{satellite child worlds are organized where radial attraction vanishes and angular return closes rationally.}}$$

The period-two component illustrates this exactly. The main cardioid has a seam at

$$\lambda = -1 = e^{2\pi i(1/2)}.$$

The resonance order is

$$q = 2.$$

The corresponding satellite world has period

$$1 \cdot 2 = 2.$$

The same point is the root of the period-two component, where the period-two multiplier is

$$\lambda = 1.$$

Thus one seam may have different internal addresses in the worlds it joins:

$$\boxed{\text{from the parent: finite angular closure;}}$$

$$\boxed{\text{from the child: parent-seam multiplier 1.}}$$

7.1 Analytic seam order versus resonance order

There are two different notions of seam complexity.

The *resonance order* is the denominator q in

$$\lambda = e^{2\pi i a/q}.$$

It records angular closure.

The *analytic seam order* records the local folding or ramification of the hidden-to-visible parameter map.

For the main cardioid, the cusp at

$$\lambda = 1$$

is analytically singular: fixed-point sheets collide. But a root of a later component may have multiplier 1 in its own multiplier coordinate without having the same cusp geometry as the main cardioid.

Thus:

$$\boxed{\text{resonance order records angular closure;}}$$

$$\boxed{\text{analytic seam order records hidden-to-visible folding.}}$$

Both are forms of finite-return structure, but they should not be conflated.

7.2 External visibility of seams

The same seam may be visible from outside the Mandelbrot set through external rays. For example, the period-two root

$$c = -\frac{3}{4}$$

is the landing point of the two external rays of angles

$$\frac{1}{3} \quad \text{and} \quad \frac{2}{3}.$$

Thus the first satellite seam is visible in three ways:

$$\boxed{\text{inside the period-one world;}}$$

$$\boxed{\text{inside the period-two world;}}$$

$$\boxed{\text{from the external escape world.}}$$

This is a prototype for the inside/outside seam structure of nonlinear worldhood.

8 Exterior return and escape reporting

The exterior of the Mandelbrot set also has radial and angular structure. Dynamical Böttcher coordinates describe exterior return; parameter Böttcher coordinates report critical escape.

There are two related Böttcher coordinates in the quadratic lane: the dynamical Böttcher coordinate and the parameter Böttcher coordinate. They should be kept separate.

8.1 Dynamical exterior return

For fixed c, the dynamical Böttcher coordinate near infinity is a conformal coordinate ϕ_c satisfying

$$\phi_c(f_c(z)) = \phi_c(z)^2$$

and

$$\phi_c(z) \sim z \qquad (z \to \infty).$$

Write

$$\phi_c(z) = e^{G_c(z)+i\Psi_c(z)}.$$

Then the identity

$$\phi_c(f_c(z)) = \phi_c(z)^2$$

gives

$$G_c(f_c(z)) = 2G_c(z),$$

and

$$\Psi_c(f_c(z)) = 2\Psi_c(z) \mod 2\pi.$$

Thus the exterior dynamical return is again radial/angular:

$$\boxed{G_c \mapsto 2G_c;}$$

$$\boxed{\Psi_c \mapsto 2\Psi_c.}$$

For the degree-d unicritical family

$$f_{d,c}(z) = z^d + c,$$

the same calculation gives

$$G_c(f_{d,c}(z)) = dG_c(z),$$

$$\Psi_c(f_{d,c}(z)) = d\Psi_c(z) \mod 2\pi.$$

Thus the degree d is both the exterior radial expansion factor and the exterior angular wrapping number.

8.2 Parameter escape reporting

For

$$c \notin \mathcal{M},$$

the critical value

$$c = f_c(0)$$

escapes. The parameter Böttcher coordinate is

$$\Phi_{\mathcal{M}}(c) := \phi_c(c).$$

It gives a conformal isomorphism

$$\Phi_{\mathcal{M}} : \mathbb{C} \setminus \mathcal{M} \to \mathbb{C} \setminus \overline{\mathbb{D}},$$

normalized by

$$\Phi_{\mathcal{M}}(c) \sim c \qquad (c \to \infty).$$

Write

$$\Phi_{\mathcal{M}}(c) = e^{G_{\mathcal{M}}(c) + i\Psi_{\mathcal{M}}(c)}.$$

Then

$$G_{\mathcal{M}}(c) = \log |\Phi_{\mathcal{M}}(c)|$$

is the Green escape function of the Mandelbrot set. Equivalently,

$$G_{\mathcal{M}}(c) = \lim_{n \to \infty} 2^{-n} \log^+ |f_c^n(c)|.$$

Since

$$c = f_c(0),$$

this may also be written as

$$G_{\mathcal{M}}(c) = \lim_{n \to \infty} 2^{-n} \log^+ |f_c^{n+1}(0)|.$$

With this normalization,

$$G_{\mathcal{M}}(c) \sim \log |c| \qquad (c \to \infty).$$

Thus:

$$G_{\mathcal{M}}(c) = 0 \quad \text{on} \quad \mathcal{M},$$

and

$$G_{\mathcal{M}}(c) > 0 \quad \text{outside} \quad \mathcal{M}.$$

The angular coordinate

$$\Psi_{\mathcal{M}}(c) = \arg \Phi_{\mathcal{M}}(c)$$

is the external address. External rays are defined by fixing this angle:

$$\mathcal{R}_{\mathcal{M}}^{\text{ext}}(\theta) = \Phi_{\mathcal{M}}^{-1}\{re^{2\pi i\theta} : r > 1\}.$$

Thus the exterior parameter world has the same radial/angular form:

$$\boxed{\text{radial coordinate} = \text{escape magnitude;}}$$

$$\boxed{\text{angular coordinate} = \text{external address.}}$$

The boundary of the Mandelbrot set is the radial-zero seam:

$$G_{\mathcal{M}} = 0.$$

This is the exterior counterpart of

$$|\lambda| = 1$$

on the boundary of a hyperbolic component.

$$\boxed{\text{interior neutral seam: } \Phi = 0; \qquad \text{exterior escape seam: } G_{\mathcal{M}} = 0.}$$

9 Singular custody

The spiral law describes local return. Nonlinear worldhood also requires custody of singular seams: points where local inverse return fails or distinguishability collapses.

For the quadratic family

$$f_c(z) = z^2 + c,$$

the derivative is

$$f_c'(z) = 2z.$$

Therefore the unique critical point is

$$0.$$

At this point,

$$f_c'(0) = 0.$$

Nearby,

$$f_c(\varepsilon) - f_c(0) = \varepsilon^2.$$

Thus

$$f_c(+\varepsilon) = f_c(-\varepsilon).$$

The critical point is the place where first-order distinguishability collapses under return.

$$\boxed{\text{critical point} = \text{local collapse of first-order distinguishability.}}$$

The critical value is

$$f_c(0) = c.$$

Thus the parameter c is the first public image of the critical seam.

The Mandelbrot set is exactly the locus where this seam remains in bounded custody:

$$\mathcal{M} = \left\{ c \in \mathbb{C} : \{f_c^n(0)\}_{n\geq 0} \text{ is bounded} \right\}.$$

The standard connectedness criterion says more generally:

Theorem 9.1 (Critical-orbit connectedness criterion, imported)**.** *Let f be a complex polynomial of degree at least two, and let*

$$K_f = \{z : \{f^n(z)\}_{n\geq 0} \text{ is bounded}\}$$

be its filled Julia set. Then K_f is connected if and only if every critical point of f has bounded forward orbit.

For the quadratic family this becomes:

$$K_c \text{ is connected} \quad \Longleftrightarrow \quad \{f_c^n(0)\}_{n\geq 0} \text{ is bounded.}$$

Equivalently,

$$c \in \mathcal{M}.$$

Thus:

$$\boxed{\text{connected nonlinear worldhood is custody of critical return.}}$$

The spiral law gives local return structure:

$$\lambda = e^{-\Phi + i\Theta}.$$

Singular custody gives the global worldhood condition:

$$\text{the singular seam must not escape.}$$

Together:

$$\boxed{\text{finite nonlinear worldhood} = \text{controlled radial/angular return} + \text{custody of singular seams.}}$$

9.1 Singular values

The critical point is not the only possible kind of singular seam in more general nonlinear return. The broader object is the singular value set: the values where inverse return fails to be locally clean.

For polynomials, singular values are critical values. For transcendental entire maps, singular values may also include asymptotic values. In the quadratic family, the singular value set is simply

$$S(f_c) = \{c\}.$$

The post-singular set is

$$P(f_c) = \overline{\{c, f_c(c), f_c^2(c), \dots\}}.$$

The Mandelbrot set is the bounded-custody locus of this one singular value.

This gives the broader General Geometry principle:

$$\boxed{\text{nonlinear worldhood is governed by custody of singular return.}}$$

The exact anchor so far is the quadratic polynomial case.

10 Singular return data and worldhood loci

The spiral law gives local return. Singular return data specify which nonlinear seams must be kept in custody for a world to form.

The Mandelbrot set is the first exact example of a more general pattern. A nonlinear world is not determined only by a return map. It is determined by a return map together with the singular seams of that map and a declared custody condition.

Definition 10.1 (Singular return datum)**.** A *singular return datum* is a package

$$\mathfrak{R} = (X, \mathfrak{P}, f_\theta, S(f_\theta), P(f_\theta), \mathcal{C}),$$

where:

- X is the return space;
- $\mathfrak{P}$ is the parameter space;
- $f_\theta : X \to X$ is a parameterized return law;
- $S(f_\theta)$ is the singular-value set of f_θ;
- $$P(f_\theta) = \overline{\bigcup_{n\geq 0} f_\theta^n(S(f_\theta))}$$

  is the post-singular set;
- $\mathcal{C}$ is a declared custody condition on $P(f_\theta)$.

Definition 10.2 (Worldhood locus)**.** Given a singular return datum $\mathfrak{R}$, its *worldhood locus* relative to the custody condition $\mathcal{C}$ is

$$\mathcal{W}_{\mathcal{C}}(\mathfrak{R}) = \{\theta \in \mathfrak{P} : P(f_\theta) \text{ satisfies } \mathcal{C}\}.$$

For polynomial Mandelbrot, the custody condition is boundedness of the critical orbit. For rational maps on the Riemann sphere, boundedness is not the primitive condition because the sphere is compact; custody may instead be hyperbolicity, postcritical stability, connectivity, or another declared condition. For transcendental maps, custody is typically phrased as control of the postsingular set.

Thus:

$$\boxed{\text{worldhood is always relative to a declared custody predicate.}}$$

10.1 One-singular-value lanes

Definition 10.3 (One-singular-value lane)**.** A family f_θ is a *one-singular-value lane* if

$$S(f_\theta) = \{s(\theta)\}$$

for a report map

$$s : \mathfrak{P} \to X.$$

In this case,

$$P(f_\theta) = \overline{\{s(\theta), f_\theta(s(\theta)), f_\theta^2(s(\theta)), \ldots\}}.$$

The worldhood problem is controlled by the orbit of one singular value.

$$\boxed{\text{one singular value} \quad \Longrightarrow \quad \text{one singular orbit controls the lane.}}$$

The quadratic Mandelbrot family is the first example:

$$f_c(z) = z^2 + c, \qquad S(f_c) = \{c\}.$$

The degree-d unicritical family

$$f_{d,c}(z) = z^d + c$$

has the same one-singular-value structure:

$$S(f_{d,c}) = \{c\}.$$

The exponential family

$$E_c(z) = e^z + c$$

also has one singular value, but of a different type: c is an asymptotic singular value rather than a critical value.

This motivates the broader distinction:

$$\boxed{\text{families may share one-singular-value form while differing by singular type.}}$$

11 Multibrot worldhood and ramification depth

The degree-d unicritical family varies the depth of the critical seam. Its public main-world cusp count is exactly the hidden critical multiplicity.

Consider the unicritical family

$$f_{d,c}(z) = z^d + c, \qquad d \geq 2.$$

Its derivative is

$$f'_{d,c}(z) = dz^{d-1}.$$

Therefore 0 is the unique critical point, with critical multiplicity

$$d - 1.$$

The critical value is

$$f_{d,c}(0) = c.$$

The degree-d connectedness locus, or Multibrot set, is

$$\mathcal{M}_d = \left\{ c \in \mathbb{C} : \{f^n_{d,c}(0)\}_{n\geq 0} \text{ is bounded} \right\}.$$

Thus $f_{d,c}$ is again a one-singular-value lane, but the singular seam has ramification depth $d - 1$.

$$\boxed{z^d + c = \text{one singular value with critical multiplicity } d - 1.}$$

11.1 The main fixed-point world

The main fixed-point world consists of parameters for which $f_{d,c}$ has an attracting fixed point. Let a be a fixed point:

$$a = a^d + c.$$

Then

$$c = a - a^d.$$

The multiplier at this fixed point is

$$\lambda = da^{d-1}.$$

Attraction means

$$|\lambda| < 1,$$

or equivalently

$$|a| < d^{-1/(d-1)}.$$

Let

$$r_d = d^{-1/(d-1)}.$$

Then the hidden fixed-point disk is

$$A_d = \{a \in \mathbb{C} : |a| < r_d\},$$

and the main fixed-point world is its image under

$$a \mapsto c = a - a^d.$$

Theorem 11.1 (Main Multibrot fixed-point world)**.** *The main fixed-point hyperbolic component of the degree-d Multibrot set is*

$$H_d^{(1)} = \{a - a^d : |a| < d^{-1/(d-1)}\}.$$

Proof. If a is a fixed point of $f_{d,c}$, then

$$a = a^d + c,$$

so

$$c = a - a^d.$$

The multiplier is

$$\lambda = da^{d-1}.$$

The fixed point is attracting exactly when

$$|\lambda| < 1,$$

that is,

$$|da^{d-1}| < 1.$$

Equivalently,

$$|a| < d^{-1/(d-1)}.$$

Conversely, for any a with

$$|a| < d^{-1/(d-1)},$$

the parameter

$$c = a - a^d$$

gives a fixed point a with attracting multiplier da^{d-1}. □

The variable a is the hidden fixed-point coordinate. The public parameter c is its surfaced report after eliminating the fixed point.

$$\boxed{\text{hidden fixed-point coordinate } a \quad \longmapsto \quad \text{public parameter } c = a - a^d.}$$

11.2 Boundary and cusp count

On the boundary of the hidden fixed-point disk, write

$$a = r_d e^{i\theta}.$$

Then

$$c(\theta) = r_d e^{i\theta} - r_d^d e^{id\theta}.$$

This is the boundary of the main fixed-point world.

Differentiate:

$$c'(\theta) = ir_d e^{i\theta} - idr_d^d e^{id\theta}.$$

Since

$$dr_d^{d-1} = 1,$$

we have

$$dr_d^d = r_d.$$

Therefore

$$c'(\theta) = ir_d e^{i\theta}\left(1 - e^{i(d-1)\theta}\right).$$

Cusps occur where

$$c'(\theta) = 0,$$

equivalently

$$e^{i(d-1)\theta} = 1.$$

There are exactly $d-1$ such points.

Theorem 11.2 (Cusp count equals critical multiplicity)**.** *The boundary of the main fixed-point world of*

$$f_{d,c}(z) = z^d + c$$

has exactly

$$d - 1$$

cusps. This number equals the critical multiplicity of the unique critical point 0*.*

Proof. The cusp condition is

$$e^{i(d-1)\theta} = 1.$$

This equation has exactly $d-1$ solutions on the circle:

$$\theta = \frac{2\pi k}{d-1}, \qquad k = 0, 1, \ldots, d-2.$$

The critical multiplicity of 0 is $d-1$, since

$$f'_{d,c}(z) = dz^{d-1}.$$

□

Thus:

$$\boxed{\#\,\mathrm{Cusps}(\partial H_d^{(1)}) = d - 1 = \text{critical multiplicity at } 0.}$$

In General Geometry language:

$$\boxed{\text{public cusp count is hidden ramification depth made visible.}}$$

Equivalently:

$$\boxed{d = \text{sheet degree}; \qquad d-1 = \text{ramification excess}; \qquad d-1 = \text{main-world cusp count.}}$$

11.3 Cusp locations

The cusp condition

$$e^{i(d-1)\theta} = 1$$

gives

$$\theta_k = \frac{2\pi k}{d-1}, \qquad k = 0, 1, \ldots, d-2.$$

The corresponding cusp points are

$$c_k = r_d e^{i\theta_k} - r_d^d e^{id\theta_k}.$$

Since

$$e^{id\theta_k} = e^{i\theta_k} e^{i(d-1)\theta_k} = e^{i\theta_k},$$

we obtain

$$c_k = (r_d - r_d^d) e^{i\theta_k}.$$

Now

$$r_d - r_d^d = d^{-1/(d-1)} - d^{-d/(d-1)} = (d-1) d^{-d/(d-1)}.$$

Therefore

$$c_k = (d-1) d^{-d/(d-1)} e^{2\pi i k/(d-1)}.$$

So the main world has $d-1$ cusp seams arranged with $(d-1)$-fold rotational symmetry.

$$\boxed{\text{critical multiplicity} \quad \longrightarrow \quad \text{rotational cusp grammar.}}$$

11.4 Fixed-point coordinate and multiplier coordinate

For $d = 2$, the fixed-point coordinate a and the multiplier coordinate are essentially the same:

$$\lambda = 2a.$$

For $d > 2$, the multiplier is

$$\lambda = d a^{d-1},$$

so the map

$$a \mapsto \lambda$$

is a $(d-1)$-fold quotient of the hidden fixed-point disk.

Thus the clean hidden coordinate for the main world is a, not λ. The multiplier disk remains the stability report, but it is not the same as the fixed-point disk when $d > 2$.

$$\boxed{\text{hidden fixed-point disk} \longrightarrow \text{main world}}$$

is given by

$$c = a - a^d,$$

while

$$\boxed{\text{hidden fixed-point disk} \longrightarrow \text{multiplier disk}}$$

is given by

$$\lambda = d a^{d-1}.$$

This distinction is invisible in the quadratic case and essential in higher degree.

$$\boxed{\text{quadratic return is the first case where fixed-point coordinate and multiplier coordinate coincide.}}$$

12 Singular-value report geometry

The singular value need not be reported directly by the public parameter. Parameter powers and logarithmic lifts are report geometries of singular return.

The family

$$f_{d,c}(z) = z^d + c$$

uses the direct report

$$s(c) = c.$$

More generally, let

$$s : \mathfrak{P} \to \mathbb{C}$$

be a singular-value report map and define

$$f_\theta(z) = z^d + s(\theta).$$

Then

$$S(f_\theta) = \{s(\theta)\}.$$

For bounded polynomial custody, the worldhood condition is simply that the singular value belongs to the Multibrot set:

$$s(\theta) \in \mathcal{M}_d.$$

Thus:

Theorem 12.1 (Report-preimage identity)**.** *For the family*

$$f_\theta(z) = z^d + s(\theta),$$

the bounded-custody worldhood locus is

$$\mathcal{W}_s = s^{-1}(\mathcal{M}_d).$$

Proof. The unique singular value is $s(\theta)$. The orbit of this singular value under

$$z \mapsto z^d + s(\theta)$$

is bounded exactly when

$$s(\theta) \in \mathcal{M}_d.$$

Therefore the parameter values producing bounded custody are exactly

$$s^{-1}(\mathcal{M}_d).$$

□

This theorem makes parameter transformations part of the formal singular-return structure. They are not merely cosmetic changes of variables; they describe how the singular value is publicly reported.

$$\boxed{\text{parameter geometry is singular-value report geometry.}}$$

12.1 Branched reports

Let

$$s(c) = c^m.$$

Then

$$f_{d,m,c}(z) = z^d + c^m$$

has bounded-custody worldhood locus

$$\mathcal{W}_{d,m} = \{c \in \mathbb{C} : c^m \in \mathcal{M}_d\}.$$

Thus the parameter plane is an m-fold branched report of the singular-value plane. The singular value is not c, but

$$c^m.$$

The two ramification depths are different:

$$d - 1 = \text{return ramification},$$

while

$$m - 1 = \text{report ramification}.$$

$$\boxed{z^d + c^m = \text{return ramification } (d-1) + \text{report ramification } (m-1).}$$

If m is composite, the report cover factors:

$$c^m = (c^a)^b \qquad (m = ab).$$

If m is prime, the report cover is indecomposable as a monomial cover degree. This is a lane-relative form of primehood:

$$\boxed{\text{parameter-primehood} = \text{indecomposable branching of the singular-value report.}}$$

12.2 Logarithmic reports

Let

$$s(u) = e^u.$$

Then

$$f_{d,u}(z) = z^d + e^u$$

has bounded-custody worldhood locus

$$\mathcal{L}_d = \{u \in \mathbb{C} : e^u \in \mathcal{M}_d\}.$$

Since

$$e^u \neq 0,$$

this is more precisely the logarithmic lift of the punctured Multibrot world:

$$\mathcal{L}_d = \exp^{-1}(\mathcal{M}_d \setminus \{0\}).$$

The center $0 \in \mathcal{M}_d$ appears as the infinite end

$$\operatorname{Re} u \to -\infty.$$

The log-lift has deck symmetry

$$u \mapsto u + 2\pi i k, \qquad k \in \mathbb{Z}.$$

Thus:

$$\boxed{\text{log-lifted worldhood} = \text{logarithmic cover of the punctured singular-value world.}}$$

Phase and sign become sheet data. For example,

$$-c = e^{u+i\pi}.$$

Thus:

$$\boxed{\text{phase is a deck coordinate of the singular-value report.}}$$

This is the report-geometric meaning of the logarithmic parameter:

$$\boxed{u = \log s}$$

as an additive coordinate on the multiplicative singular-value report.

12.3 Report ramification versus return ramification

The report map and the return map carry different types of ramification.

For

$$f_{d,m,c}(z) = z^d + c^m,$$

the return map has critical multiplicity

$$d - 1$$

at the dynamical critical point 0. The report map

$$c \mapsto c^m$$

has ramification multiplicity

$$m - 1$$

at the parameter point $c = 0$.

Thus:

$$\boxed{\text{return ramification belongs to the dynamical plane;}}$$

$$\boxed{\text{report ramification belongs to the parameter/report plane.}}$$

The two may interact, but they are not the same object.

This distinction is important for General Geometry: public parameters may carry their own branching before the singular value reaches the return law.

$$\boxed{\text{what returns} \neq \text{how its singular value is reported.}}$$

13 Pole-mediated and Laurent return

Negative powers and Laurent maps move singular return from the polynomial plane to the Riemann sphere. Poles then mediate critical orbit structure.

The polynomial families above live naturally on the complex plane. Negative powers require the Riemann sphere

$$\widehat{\mathbb{C}}.$$

The key new feature is that ∞ becomes part of the return geometry.

13.1 Negative powers

Consider

$$R_{d,c}(z) = z^{-d} + c.$$

As a rational map on $\widehat{\mathbb{C}}$,

$$R_{d,c}(z) = \frac{1 + cz^d}{z^d}.$$

Its degree is d.

At

$$z = 0,$$

there is a pole of local degree d. Thus 0 is a critical point of multiplicity

$$d - 1$$

with critical value

$$\infty.$$

At

$$z = \infty,$$

use the local coordinate

$$w = \frac{1}{z}.$$

Then

$$R_{d,c}(1/w) = w^d + c.$$

Thus ∞ is a critical point of multiplicity

$$d - 1$$

with critical value

$$c.$$

The critical orbit structure includes

$$0 \mapsto \infty \mapsto c.$$

The total critical multiplicity is

$$(d - 1) + (d - 1) = 2d - 2,$$

as required for a degree-d rational map.

Thus:

$$\boxed{z^{-d} + c = \text{pole-mediated critical return on the Riemann sphere.}}$$

The word "pole" should be read carefully. In rational dynamics, singular values are critical values; poles are preimages of ∞. The pole matters because it places ∞ into the critical orbit structure.

$$\boxed{\text{negative powers introduce pole-mediated critical orbits.}}$$

13.2 Laurent worlds

Consider the Laurent family

$$F_{d,m,c}(z) = z^d + \frac{c}{z^m}, \qquad d, m \geq 1.$$

As a rational map,

$$F_{d,m,c}(z) = \frac{z^{d+m} + c}{z^m}.$$

For

$$c \neq 0,$$

its degree is

$$d + m.$$

The derivative is

$$F'_{d,m,c}(z) = dz^{d-1} - mcz^{-m-1}.$$

Finite critical points satisfy

$$dz^{d+m} = mc.$$

For

$$c \neq 0,$$

there are

$$d + m$$

simple finite critical points.

At

$$z = 0,$$

there is a pole of local degree m, hence critical multiplicity

$$m - 1.$$

At

$$z = \infty,$$

the leading behavior is

$$z^d,$$

so there is critical multiplicity

$$d - 1.$$

The total critical multiplicity is

$$(d + m) + (m - 1) + (d - 1) = 2(d + m) - 2,$$

which agrees with Riemann–Hurwitz for a rational map of degree $d+m$.

Thus:

$$\boxed{z^d + \frac{c}{z^m} = \text{finite critical constellation} + \text{zero-pole seam} + \text{infinity seam.}}$$

The pole and infinity seams both have critical value

$$\infty,$$

while the finite critical points generally produce a finite critical-value constellation.

13.3 Finite critical-value collisions

At a finite critical point,

$$dz^{d+m} = mc.$$

Therefore

$$cz^{-m} = \frac{d}{m} z^d.$$

So

$$F_{d,m,c}(z) = z^d + \frac{d}{m} z^d = \frac{d+m}{m} z^d.$$

The finite critical points are the $(d+m)$-th roots of

$$\frac{m}{d} c.$$

The map from these critical points to finite critical values is

$$z \mapsto z^d.$$

On the set of $(d+m)$-th roots, the number of distinct images under $z \mapsto z^d$ is

$$\frac{d+m}{\gcd(d, d+m)} = \frac{d+m}{\gcd(d, m)}.$$

Theorem 13.1 (Laurent critical-value collision count)**.** *For*

$$F_{d,m,c}(z) = z^d + \frac{c}{z^m}$$

with $c \neq 0$*, the number of distinct finite critical values is*

$$\frac{d+m}{\gcd(d, m)}.$$

Each distinct finite critical value is hit by

$$\gcd(d, m)$$

finite critical points.

Proof. The finite critical points are the solutions of

$$z^{d+m} = \frac{m}{d}c.$$

They form a torsor over the group of $(d+m)$-th roots of unity. The finite critical value depends on z through z^d, since

$$F_{d,m,c}(z) = \frac{d+m}{m}z^d$$

at a finite critical point.

The image of the map

$$\zeta \mapsto \zeta^d$$

from the $(d+m)$-th roots of unity has size

$$\frac{d+m}{\gcd(d, d+m)}.$$

Since

$$\gcd(d, d+m) = \gcd(d, m),$$

the number of distinct finite critical values is

$$\frac{d+m}{\gcd(d, m)}.$$

The fiber size is the kernel size,

$$\gcd(d, d+m) = \gcd(d, m).$$

□

Thus:

$$\boxed{\gcd(d, m) = \text{finite critical-value collision multiplicity in the Laurent lane.}}$$

This is the cleanest arithmetic shadow in the pole-mediated return family.

13.4 Return roles in the Laurent lane

The Laurent family separates several roles that coincide in simpler polynomial lanes:

$$d = \text{degree of positive-power return at infinity,}$$

$$m = \text{pole order at zero,}$$

$$d+m = \text{rational degree and finite critical count,}$$

$$\gcd(d, m) = \text{critical-value collision multiplicity.}$$

Thus:

$$\boxed{\text{Laurent return separates sheet count, pole order, critical count, and collision grammar.}}$$

This is an important General Geometry lesson: an integer does not carry one universal meaning. Its meaning is typed by the return lane in which it appears.

14 Exponential singular return

The exponential family replaces finite critical ramification with an asymptotic singular value and infinitely many inverse sheets.

Consider

$$E_c(z) = e^z + c.$$

Then

$$E_c'(z) = e^z \neq 0.$$

Thus E_c has no critical points.

However, as

$$\operatorname{Re} z \to -\infty,$$

we have

$$e^z + c \to c.$$

So c is an asymptotic singular value.

The inverse branches are

$$z = \log(w - c) + 2\pi i k, \qquad k \in \mathbb{Z}.$$

Thus inverse return is infinite-sheeted.

$$\boxed{e^z + c = \text{one asymptotic singular value with infinite inverse sheets.}}$$

The post-singular set is controlled by the orbit

$$c, \quad e^c + c, \quad e^{e^c + c} + c, \quad \ldots$$

There is no direct polynomial filled-Julia connectedness theorem of the same form as for polynomials. The safe General Geometry statement is:

$$\boxed{\text{exponential worldhood is governed by custody of an asymptotic singular value.}}$$

Thus the exponential family is a one-singular-value lane, but not a critical-ramification lane. It is an infinite-sheeted singular-return lane.

$$\boxed{\text{finite ramification} \quad \rightsquigarrow \quad \text{infinite-sheeted asymptotic return.}}$$

14.1 Deck structure

The inverse branches

$$z = \log(w - c) + 2\pi i k$$

are indexed by

$$k \in \mathbb{Z}.$$

The deck shift is

$$z \mapsto z + 2\pi i.$$

Thus the exponential lane carries a built-in infinite angular sheet structure.

This contrasts with the finite monomial lane

$$z^d + c,$$

where inverse return has finitely many sheets and critical ramification. The exponential lane has no critical points, but its inverse return has infinitely many logarithmic sheets.

Thus:

$$\boxed{\text{monomial return} = \text{finite critical ramification;}}$$

$$\boxed{\text{exponential return} = \text{infinite-sheeted asymptotic report.}}$$

15 Typed primehood in singular-return lanes

Primehood is not one undifferentiated property. In singular-return geometry, different integers measure different kinds of non-splitting.

Several integers appear naturally in singular-return lanes:

$$d, \qquad d-1, \qquad m, \qquad d+m, \qquad \gcd(d,m), \qquad p, \qquad q.$$

They should not be collapsed into one role. Each records a different aspect of return, ramification, report, period, or resonance.

integer	role
d	cover degree / sheet count of z^d
$d-1$	critical multiplicity / ramification excess / Multibrot cusp count
m	report-cover degree in c^m or pole order in $z^d + c/z^m$
$d+m$	degree and finite critical count in the Laurent family
$\gcd(d,m)$	finite critical-value collision multiplicity in the Laurent family
p	cycle period
q	neutral resonance order / finite angular closure order

Thus:

$$\boxed{\text{primehood in singular-return geometry is typed.}}$$

For example, a prime d means that the monomial cover

$$z \mapsto z^d$$

is indecomposable as a monomial cover degree. If

$$d = ab,$$

then

$$z^d = (z^a)^b,$$

so the cover degree factors.

A prime q, by contrast, means that the finite angular closure order

$$e^{2\pi ip/q}$$

is indecomposable as a phase-closure denominator.

A prime m in a branched report

$$s(c) = c^m$$

means that the singular-value report is indecomposable as a monomial report cover. In the Laurent family, the same letter m measures pole order instead.

The General Geometry reading is:

$$\boxed{\text{primehood means non-splitting relative to a specified lane and codebook.}}$$

The lane must be declared before primehood has a precise meaning.

16 Singular-return taxonomy

The spiral law supplies the local return normal form. Singular-return taxonomy classifies the seams whose custody makes nonlinear return into worldhood.

The families considered above form the first singular-return atlas.

Family	**Singular / report structure**	**General Geometry role**
$z^2 + c$	One simple critical point 0, critical value c.	Quadratic Mandelbrot lane: bounded custody of one simple critical seam.
$z^d + c$	One critical point 0 of multiplicity $d-1$, critical value c.	Multibrot lane: ramification depth. Main-world cusp count equals critical multiplicity.
$z^d + c^m$	Dynamic critical multiplicity $d-1$; singular value reported through c^m.	Branched report lane. Return ramification and report ramification separate.
$z^d + e^u$	Dynamic critical multiplicity $d-1$; singular value reported through e^u.	Log-lifted report lane. Phase and sign become deck data; 0 becomes an infinite end.
$z^{-d} + c$	Rational map on $\widehat{\mathbb{C}}$. Critical points at 0 and ∞, both of multiplicity $d-1$, with $0 \mapsto \infty \mapsto c$.	Pole-mediated critical return. Negative powers introduce critical orbits through ∞.
$z^d + \dfrac{c}{z^m}$	Rational map of degree $d+m$. Finite critical constellation plus pole seam at 0 and critical behavior at ∞.	Laurent worldhood. Critical-value collision multiplicity is $\gcd(d, m)$.
$e^z + c$	No critical points; one asymptotic singular value c; infinitely many inverse branches.	Exponential singular return. Finite ramification is replaced by infinite-sheeted asymptotic custody.

This taxonomy should be read as a hierarchy of singular-return types:

finite critical return $\rightsquigarrow$ report-branched critical return $\rightsquigarrow$ pole-mediated return $\rightsquigarrow$ asymptotic singul

The shared structure is not that all these families have the same worldhood condition. They do not. The shared structure is that nonlinear worldhood is governed by custody of singular return.

$$\boxed{\text{different singular types require different custody predicates.}}$$

For polynomials, bounded critical-orbit custody is the classical connectedness condition. For rational maps on the sphere, custody is formulated through postcritical stability, hyperbolicity, connectivity, or another declared dynamical predicate. For transcendental maps, custody is postsingular control.

Thus the formal object remains:

$$\mathcal{W}_{\mathcal{C}} = \{\theta \in \mathfrak{P} : P(f_\theta) \text{ satisfies } \mathcal{C}\}.$$

17 Synthesis: the law of finite return

The scalar spiral law, higher-dimensional polar return, carrier continuation, Fibonacci projective return, Mandelbrot critical return, and Omnibrot extensions are witnesses of one architecture: finite return has radial fate and angular continuation, and nonlinear worldhood requires custody of singular seams.

The return structure appears in several lanes.

17.1 Scalar return

A nonzero one-dimensional return mode has multiplier

$$\lambda = e^{-\Phi+i\Theta}.$$

The radial return exponent is

$$\Phi = -\log|\lambda|,$$

and the angular return class is

$$\Theta = \arg(\lambda) \in \mathbb{R}/2\pi\mathbb{Z}.$$

In log-polar coordinates

$$u = e^{R+i\varphi},$$

return acts as

$$R \mapsto R - \Phi, \qquad \varphi \mapsto \varphi + \Theta.$$

Thus:

$$\boxed{\text{scalar finite return} = \text{radial fate} + \text{angular continuation.}}$$

17.2 Higher-dimensional return

For an invertible finite-dimensional return operator

$$A : V \to V,$$

polar decomposition gives

$$A = U_A e^{-H_A},$$

where U_A is unitary and H_A is self-adjoint.

Thus:

$$\boxed{\text{finite-dimensional return} = \text{radial operator} + \text{angular transport.}}$$

If A is normal, then

$$A = \sum_j e^{-\Phi_j + i\Theta_j} \Pi_j,$$

so return decomposes into independent scalar spiral modes.

If A is nonnormal, radial return and angular transport fail to share a common orthogonal modal frame. The defect

$$\mathfrak{S}(A) = \|A^*A - AA^*\|_{\mathrm{HS}}$$

detects this failure:

$$\mathfrak{S}(A) = 0 \quad \Longleftrightarrow \quad A \text{ is normal.}$$

Thus:

$$\boxed{\text{nonnormality} = \text{finite-return shear.}}$$

Exterior powers carry return to forms:

$$\bigwedge^k A = (\bigwedge^k U_A) e^{-H_A^{(k)}}.$$

The k-form radial exponents are sums

$$\Phi_{i_1} + \cdots + \Phi_{i_k}.$$

The determinant gives top-volume return:

$$-\log|\det A| = \operatorname{tr} H_A.$$

Thus:

$$\boxed{\text{vectors, areas, volumes, and forms all carry finite-return data.}}$$

If A is noninvertible, some first-order distinguishability collapses. For a nonlinear return law f, this is detected by noninvertibility of

$$Df_x.$$

In equal dimensions:

$$\det Df_x = 0$$

is the top-volume collapse condition.

17.3 Carrier lane

On a spectral mode,

$$e^{-(\sigma+is)W}\psi = e^{-\sigma\omega} e^{-is\omega}\psi.$$

Thus

$$\Phi_\omega = \sigma\omega, \qquad \Theta_\omega = -s\omega.$$

Poisson holding and exact continuation are the real and imaginary faces of the first-order carrier family.

The heat semigroup

$$e^{-rW^2}$$

is the radial rebuild flow generated by the burden operator

$$L = W^2.$$

Thus:

$$\boxed{\text{first-order carrier return} = \text{Poisson holding} + \text{exact continuation.}}$$

17.4 Golden lane

The golden projective map

$$T(r) = 1 + \frac{1}{r}$$

has attracting fixed point

$$\varphi$$

and projective multiplier

$$T'(\varphi) = -\varphi^{-2} = e^{-2\log\varphi + i\pi}.$$

Thus golden projective return has radial exponent

$$2\log\varphi$$

and angular class

$$\pi.$$

The exact cross-ratio law

$$\frac{T^n(r) - \varphi}{T^n(r) - \varphi'} = \left(-\varphi^{-2}\right)^n \frac{r - \varphi}{r - \varphi'}$$

shows projective spiral stabilization toward the golden line.

The Fibonacci identity

$$\mathsf{F}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$$

then makes Fibonacci ratios public integer shadows of that stabilization.

Thus:

$$\boxed{\text{Fibonacci} = \text{public integer shadow of golden projective spiral return.}}$$

17.5 Quadratic Mandelbrot lane

A nonlinear periodic role has multiplier

$$\lambda = (f_c^p)'(z_0).$$

Writing

$$\lambda = e^{-\Phi + i\Theta}$$

separates attraction, neutrality, and expansion. For fixed points of

$$f_c(z) = z^2 + c,$$

eliminating the hidden fixed point gives

$$c = \frac{\lambda}{2} - \frac{\lambda^2}{4}.$$

The open main cardioid is therefore the visible parameter shadow of the fixed-point return disk

$$|\lambda| < 1.$$

In this nonlinear lane, local return is not enough. The critical seam

$$0$$

must remain in custody:

$$c \in \mathcal{M} \quad \Longleftrightarrow \quad \{f_c^n(0)\}_{n \geq 0} \text{ is bounded.}$$

Thus:

$$\boxed{\text{Mandelbrot worldhood} = \text{controlled nonlinear return with bounded critical custody.}}$$

17.6 Multibrot and ramification depth

For

$$f_{d,c}(z) = z^d + c,$$

the critical point 0 has multiplicity

$$d - 1.$$

The main fixed-point world is the image of

$$|a| < d^{-1/(d-1)}$$

under

$$c = a - a^d.$$

Its boundary has exactly

$$d - 1$$

cusps.

Thus:

$$\boxed{\#\,\text{Cusps} = d - 1 = \text{critical multiplicity.}}$$

The public cusp count records hidden ramification depth.

17.7 Report geometry

For

$$f_\theta(z) = z^d + s(\theta),$$

the singular value is

$$s(\theta).$$

The bounded-custody locus is

$$\mathcal{W}_s = s^{-1}(\mathcal{M}_d).$$

Thus singular-value report maps are part of the geometry. The report

$$s(c) = c^m$$

adds report ramification. The report

$$s(u) = e^u$$

gives a logarithmic cover with deck symmetry

$$u \mapsto u + 2\pi i k.$$

Thus:

$$\boxed{\text{parameter geometry is singular-value report geometry.}}$$

17.8 Pole and asymptotic return

Negative powers move the lane to the Riemann sphere:

$$z^{-d} + c$$

has pole-mediated critical orbits through

$$0 \mapsto \infty \mapsto c.$$

Laurent maps

$$z^d + \frac{c}{z^m}$$

combine finite critical constellations with pole seams. Their finite critical values collide with multiplicity

$$\gcd(d, m).$$

The exponential family

$$e^z + c$$

has one asymptotic singular value c and infinitely many inverse sheets.

Thus:

$$\boxed{\text{singular-return worlds are classified by singular type, ramification depth, and report geometry.}}$$

17.9 The law

The common formal content is:

$$\boxed{\text{every nonzero one-dimensional return mode has radial fate and angular law.}}$$

The higher-dimensional content is:

$$\boxed{\text{finite-dimensional return has radial operator, angular transport, and form-level return.}}$$

The nonlinear extension is:

$$\boxed{\text{worldhood requires custody of singular seams.}}$$

The singular-return extension is:

$$\boxed{\text{public shadows record the ramification and reporting of return.}}$$

Combining these:

$$\boxed{\text{finite worldhood is controlled radial/angular return together with custody of singular seams.}}$$

Or, in the shortest form:

$$\boxed{\text{to return finitely is to spiral.}}$$

The word *spiral* names the normal form

$$R \mapsto R - \Phi, \qquad \varphi \mapsto \varphi + \Theta.$$

A finite world is not merely a static object. It is a return regime. Its modes must have controlled radial fate and angular continuation. Its vectors, forms, loops, and public reports carry return data. In nonlinear lanes, its singular seams must not escape. In singular-return extensions, its public shapes record the depth and report geometry of those seams.

$$\boxed{\text{A finite world is a held return regime whose modes spiral and whose seams remain in custody.}}$$

18 Conclusion

A complex return multiplier contains two unavoidable pieces:

$$\lambda = e^{-\Phi + i\Theta}.$$

This is the first nontrivial local form of finite return: radial fate plus angular continuation.

Finite-dimensional return carries the same law through polar decomposition:

$$A = U_A e^{-H_A}.$$

Exterior powers carry it to areas, volumes, and forms:

$$\bigwedge^k A = (\bigwedge^k U_A) e^{-H_A^{(k)}}.$$

The determinant gives total volume return:

$$-\log|\det A| = \operatorname{tr} H_A.$$

Noninvertibility is first-order collapse of distinguishability.

The carrier family carries the scalar form through

$$e^{-(\sigma + is)W}.$$

The Fibonacci lane carries it through golden projective stabilization:

$$-\varphi^{-2} = e^{-2\log\varphi + i\pi}.$$

The Mandelbrot lane carries it through nonlinear periodic multipliers:

$$\lambda = (f_c^p)'(z_0).$$

The nonlinear lane adds a further condition: singular custody. For

$$f_c(z) = z^2 + c,$$

the critical point is 0, and the Mandelbrot set is exactly the locus where its orbit remains bounded:

$$\mathcal{M} = \{c : \{f_c^n(0)\}_{n\geq 0} \text{ is bounded}\}.$$

Thus finite nonlinear worldhood is not only controlled return. It is controlled return with the critical seam kept in custody.

The degree-d Multibrot lane varies the depth of this seam:

$$\#\,\text{Cusps} = d - 1.$$

Report maps vary how the singular value becomes public:

$$\mathcal{W}_s = s^{-1}(\mathcal{M}_d).$$

Pole-mediated and Laurent return add critical orbits through ∞, and exponential return replaces critical ramification with asymptotic singular custody.

The final principle is:

$\boxed{\textbf{Spiral Law of Finite Return:}}$

$\boxed{\text{Every nonzero one-dimensional return mode decomposes into a radial return exponent and an angular return class.}}$

The higher-dimensional principle is:

$\boxed{\text{Every invertible finite-dimensional return operator decomposes into radial return and angular transport.}}$

The nonlinear principle is:

$\boxed{\text{finite nonlinear worldhood is controlled radial/angular return together with custody of singular seams.}}$

The corresponding singular-return doctrine is:

$\boxed{\text{singular-return worlds are classified by singular type, ramification depth, and report geometry.}}$

Fibonacci, Mandelbrot, Multibrot, Laurent, and exponential return are not the same lane, but they expose the same architecture at different depths.

$\boxed{\text{The spiral is the normal form of finite return.}}$

$\boxed{\text{Singular custody is the nonlinear condition of worldhood.}}$

$\boxed{\text{Public world-shapes record the ramification and reporting of return.}}$

A Claim ledger

This appendix records the status of the main claims.

Exact calculations

Every nonzero complex multiplier has the form

$$\lambda = e^{-\Phi + i\Theta}, \qquad \Phi = -\log|\lambda|, \qquad \Theta = \arg(\lambda) \in \mathbb{R}/2\pi\mathbb{Z}.$$

Every invertible finite-dimensional return operator has polar form

$$A = U_A e^{-H_A},$$

where U_A is unitary and H_A is self-adjoint.

For a normal return operator,

$$A = \sum_j e^{-\Phi_j + i\Theta_j} \Pi_j.$$

The nonnormal shear defect

$$\mathfrak{S}(A) = \|A^*A - AA^*\|_{\mathrm{HS}}$$

vanishes exactly when A is normal.

Exterior-power return satisfies

$$\bigwedge^k A = (\bigwedge^k U_A) e^{-H_A^{(k)}}.$$

The determinant return satisfies

$$-\log|\det A| = \operatorname{tr} H_A.$$

On a spectral mode

$$W\psi = \omega\psi,$$

one has

$$e^{-(\sigma + is)W}\psi = e^{-\sigma\omega} e^{-is\omega}\psi.$$

The golden projective map

$$T(r) = 1 + \frac{1}{r}$$

has

$$T'(\varphi) = -\varphi^{-2} = e^{-2\log\varphi + i\pi}.$$

The cross-ratio identity is

$$\frac{T^n(r) - \varphi}{T^n(r) - \varphi'} = \left(-\varphi^{-2}\right)^n \frac{r - \varphi}{r - \varphi'}.$$

The Fibonacci vector identity is

$$\mathsf{F}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}.$$

For

$$f_c(z) = z^2 + c,$$

the unique critical point is

$$0,$$

and the critical value is

$$f_c(0) = c.$$

A fixed point z_* has multiplier

$$\lambda = 2z_*.$$

Eliminating z_* gives

$$c = \frac{\lambda}{2} - \frac{\lambda^2}{4}.$$

The fixed-point equation

$$z^2 - z + c = 0$$

has discriminant

$$\Delta = 1 - 4c.$$

For

$$f_{d,c}(z) = z^d + c,$$

the unique critical point 0 has multiplicity $d-1$, and the main fixed-point world is

$$H_d^{(1)} = \{a - a^d : |a| < d^{-1/(d-1)}\}.$$

Its boundary has exactly

$$d - 1$$

cusps.

For

$$f_\theta(z) = z^d + s(\theta),$$

the bounded-custody locus is

$$\mathcal{W}_s = s^{-1}(\mathcal{M}_d).$$

For

$$F_{d,m,c}(z) = z^d + \frac{c}{z^m},$$

with $c \neq 0$, the number of distinct finite critical values is

$$\frac{d+m}{\gcd(d,m)},$$

and each is hit by $\gcd(d,m)$ finite critical points.

Imported exact facts

Polar decomposition holds for finite-dimensional invertible operators on complex Hilbert spaces.

For a complex polynomial f of degree at least two, the filled Julia set

$$K_f = \{z : \{f^n(z)\}_{n\geq 0} \text{ is bounded}\}$$

is connected if and only if every critical orbit is bounded.

For the quadratic family, this gives

$$c \in \mathcal{M} \iff \{f_c^n(0)\}_{n\geq 0} \text{ is bounded}.$$

Each hyperbolic component of the quadratic Mandelbrot set is conformally parameterized by the multiplier disk of its attracting cycle.

The dynamical Böttcher coordinate satisfies

$$\phi_c(f_c(z)) = \phi_c(z)^2.$$

The parameter Böttcher coordinate is

$$\Phi_{\mathcal{M}}(c) = \phi_c(c),$$

and the Green function satisfies

$$G_{\mathcal{M}}(c) = \log|\Phi_{\mathcal{M}}(c)| = \lim_{n\to\infty} 2^{-n}\log^+|f_c^n(c)| = \lim_{n\to\infty} 2^{-n}\log^+|f_c^{n+1}(0)|.$$

For a rational map of degree N, the total critical multiplicity on the Riemann sphere is

$$2N-2.$$

For transcendental entire maps, singular values may include asymptotic values in addition to critical values.

General Geometry interpretations

$$\boxed{\text{radial return exponent} = \text{radial fate of finite return.}}$$

$$\boxed{\text{angular return class} = \text{angular continuation data of finite return.}}$$

$$\boxed{\text{polar decomposition} = \text{higher-dimensional radial/angular return.}}$$

$$\boxed{\text{nonnormality} = \text{finite-return shear.}}$$

$$\boxed{\text{exterior powers} = \text{return of areas, volumes, and forms.}}$$

$$\boxed{\text{Fibonacci ratios} = \text{public integer shadows of golden projective spiral stabilization.}}$$

$$\boxed{\text{open main cardioid} = \text{visible parameter shadow of the fixed-point return disk.}}$$

$$\boxed{\text{critical point} = \text{local collapse of first-order distinguishability.}}$$

$$\boxed{\text{bounded critical orbit} = \text{custody of singular return.}}$$

$$\boxed{\text{Multibrot cusp count} = \text{hidden ramification depth made visible.}}$$

$$\boxed{\text{parameter geometry} = \text{singular-value report geometry.}}$$

$$\boxed{\text{finite worldhood} = \text{controlled radial/angular return plus custody of singular seams.}}$$

$$\boxed{\text{singular-return worlds} = \text{worlds classified by singular type, ramification depth, and report geometry.}}$$

Guardrails

The scalar spiral law is a one-dimensional normal form.

The polar return theorem is finite-dimensional and invertible. Noninvertible operators lie on the singular boundary of the theorem and must be treated through collapse of distinguishability.

The value $\lambda = 0$ is a degenerate first-order case. It indicates superattracting or higher-order ramified return rather than a nonzero multiplier spiral.

The value $\lambda = 1$ is a neutral first-order seam. Its behavior is governed by higher-order parabolic terms.

For rational maps on the Riemann sphere, boundedness is not the primary custody condition. Custody must be declared as postcritical stability, hyperbolicity, connectivity, or another appropriate dynamical predicate.

For exponential and other transcendental families, the polynomial critical-orbit connectedness theorem does not directly apply. The relevant object is postsingular control.